7.j) $y''' - 3iy'' - 3y' + iy = 0$

Soln:

The c.p is $r^3 - 3ir^2 - 3r + i = 0$

The roots are

$r = i, i, i$

The soln of $\phi(x)$ has the form

$\phi(x) = (c_1 + c_2 x + c_3 x^2) e^{ix}$

	1	$-3i$	-3
i	0	i	...
	1	$-2i$	-1
i	0	i	1
	1	$-i$	0

⑦ k) $y^{(100)} + 100y = 0$.

8.a) Find the soln of ϕ of the initial value problem.

$y''' + y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0$

Soln:

Given eqn is $y''' + y = 0$

The c.p is $r^3 + 1 = 0$

$r^3 + 1^3 = 0$

$(r + 1)(r^2 - r + 1) = 0$

$a = 1, b = -1, c = 1$

$r + 1 = 0, \quad r^2 - r + 1 = 0$

$= \frac{1 \pm \sqrt{1-4}}{2}$

$r = -1, \quad r = \frac{1 \pm i\sqrt{3}}{2}$

$= \frac{1 \pm i\sqrt{3}}{2}$

$r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

The soln of $y(x)$ has the form

$y(x) = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

$y(x) = c_1 e^{-x} + c_2 e^{\frac{x}{2}} \cos\frac{\sqrt{3}}{2}x + c_3 e^{x/2} \sin\frac{\sqrt{3}}{2}x \rightarrow ①$

Given that $y(0) = 0$

① $\Rightarrow \quad c_1 + c_2 = 0 \quad \rightarrow ②$

Diff ① w.r.t x

$y'(x) = -c_1 e^{-x} + c_2\left[e^{\frac{x}{2}}\left(-\sin\frac{\sqrt{3}}{2}x\right)\cdot\frac{\sqrt{3}}{2} + e^{x/2}\cdot\frac{1}{2}\cos\frac{\sqrt{3}}{2}x\right] +$

$c_3\left[e^{x/2}\left(\cos\frac{\sqrt{3}}{2}x\right)\frac{\sqrt{3}}{2} + e^{x/2}\cdot\frac{1}{2}\sin\frac{\sqrt{3}}{2}x\right]$

$y'(0) = -c_1 + \frac{c_2}{2} + \frac{\sqrt{3}}{2}c_3$

$\because y'(0) = 1$

$-2c_1 + c_2 + \sqrt{3}c_3 = 2 \quad \rightarrow ③$

(16)

$$y''(x) = c_1 e^{x} + c_2 \left\{ -\frac{\sqrt{3}}{2}\left[e^{x/2}\left(\cos\frac{\sqrt{3}}{2}x\right)\frac{\sqrt{3}}{2} + e^{x/2}\cdot\frac{1}{2}\left(-\sin\frac{\sqrt{3}}{2}x\right)\right] + \frac{1}{2}\left[e^{x/2}\left(-\sin\frac{\sqrt{3}}{2}x\right)\cdot\frac{\sqrt{3}}{2} + e^{x/2}\,\frac{1}{2}\cos\frac{\sqrt{3}}{2}x\right]\right\}$$

$$+ c_3\left\{\frac{\sqrt{3}}{2}\left[e^{x/2}\left(-\sin\frac{\sqrt{3}}{2}x\right)\cdot\frac{\sqrt{3}}{2} + e^{x/2}\cdot\frac{1}{2}\cos\frac{\sqrt{3}}{2}x\right] + \frac{1}{2}\left[e^{x/2}\left(\cos\frac{\sqrt{3}}{2}x\right)\cdot\frac{\sqrt{3}}{2} + e^{x/2}\cdot\frac{1}{2}\sin\frac{\sqrt{3}}{2}x\right]\right\}$$

$$y''(0) = c_1 + c_2\left(-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{4}\right) + c_3\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$$

$$0 = c_1 + c_2\left(-\frac{3}{4} + \frac{1}{4}\right) + c_3\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)$$

$$c_1 + \left(-\frac{1}{2}c_2\right) + \frac{2\sqrt{3}}{4}c_3 = 0$$

$$c_1 - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3 = 0 \rightarrow (4)$$

(3) $\Rightarrow$ $-2c_1 + c_2 + \sqrt{3}c_3 = 2$

(4)$\times 2 \Rightarrow$ $2c_1 - c_2 + \sqrt{3}c_3 = 0$

$(-)\quad(+)\quad(-)$

$$-4c_1 + 2c_2 = 2$$

$$-2c_1 + c_2 = 1 \rightarrow (5)$$

(2) $\Rightarrow$ $c_1 + c_2 = 0$

(5) $\Rightarrow$ $-2c_1 + c_2 = 1$

$(+)\quad(-)\quad -$

$$3c_1 = -1$$

$$c_1 = -1/3$$

Sub in (2)

$$-\frac{1}{3} + c_2 = 0$$

$$c_2 = \frac{1}{3}$$

c_1, c_2 sub in (3)

$$-2c_1 + c_2 + \sqrt{3}c_3 = 2$$

$$-2\left(-\frac{1}{3}\right) + \frac{1}{3} + \sqrt{3}c_3 = 2$$

$$\frac{2}{3} + \frac{1}{3} + \sqrt{3}c_3 = 2$$

$$1 + \sqrt{3}c_3 = 2$$

$$\sqrt{3}c_3 = 1$$

$$c_3 = \frac{1}{\sqrt{3}}$$

These values sub in (1) $\Rightarrow$ $y(x) = -\frac{1}{3}e^{x} + \frac{1}{3}e^{x/2}\cos\frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}}e^{x/2}\sin\frac{\sqrt{3}}{2}x$.

8.b) $y^{(4)} + 16y = 0$ Compute the Wronskian of four linearly independent solutions for the above equation. Compute the solution ϕ of this equation which satisfies $\phi(0) = 1$, $\phi'(0) = 0$, $\phi''(0) = 0$, $\phi'''(0) = 0$.

Soln:

Given eqn is $y^{(4)} + 16y = 0$

The C.P is

$$r^4 + 16 = 0$$

$$r^4 + 2^4 = 0$$

$\div r^2$

$$r^2 + \frac{2^4}{r^2} = 0 \quad \rightarrow (1)$$

Let $t = r + \frac{2^2}{r}$

$$t^2 = \left(r + \frac{2^2}{r}\right)^2 = r^2 + \frac{2^4}{r^2} + \frac{8r}{r}$$

$$= r^2 + \frac{2^4}{r^2} + 8$$

$$t^2 - 8 = r^2 + \frac{2^4}{r^2}$$

$$t^2 - 8 = 0 \quad \text{(from (1))}$$

$$t^2 = 8 \Rightarrow t = \pm 2\sqrt{2}$$

Put $t = 2\sqrt{2}$

$$r + \frac{2^2}{r} = 2\sqrt{2}$$

$$r^2 + 4 = 2\sqrt{2}\, r$$

$$r^2 - 2\sqrt{2}\, r + 4 = 0$$

$$r = \frac{2\sqrt{2} \pm \sqrt{8-16}}{2}$$

$$= \frac{2\sqrt{2} \pm \sqrt{-8}}{2}$$

$$= \frac{2\sqrt{2} \pm i 2\sqrt{2}}{2}$$

$$= \sqrt{2} \pm i\sqrt{2}$$

$t = -2\sqrt{2}$

$$r + \frac{2^2}{r} = -2\sqrt{2}$$

$$r^2 + 4 = -2\sqrt{2}\, r$$

$$r^2 + 2\sqrt{2}\, r + 4 = 0$$

$$r = \frac{-2\sqrt{2} \pm \sqrt{8-16}}{2}$$

$$= \frac{-2\sqrt{2} \pm \sqrt{-8}}{2}$$

$$= \frac{-2\sqrt{2} \pm i 2\sqrt{2}}{2}$$

$$= -\sqrt{2} \pm i\sqrt{2}$$

The Soln $\phi(x)$ has the form

$$\phi(x) = e^{\sqrt{2}x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + e^{-\sqrt{2}x}(c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x)$$

8.c $y''' - 4y' = 0$.

Soln:

Given eqn is $y''' - 4y' = 0$

The c.p is $r^3 - 4r = 0$

$r(r^2 - 4) = 0$

$r = 0, \quad r^2 - 2^2 = 0$

$(r-2)(r+2) = 0$

$r = 2, -2$

The roots are $r = 0, 2, -2$

The soln φ has the form

$\varphi(x) = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$

$\varphi(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

8.d) Are the following sets of functions defined on $-\infty < x < \infty$ linearly independent or dependent there?

$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \quad \varphi_3(x) = x^3$

Soln:

The Wronskian of $\varphi_1, \varphi_2, \varphi_3$ is

$$W(\varphi_1, \varphi_2, \varphi_3) = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$= 1(2-0) - x(0-0) + x^2(0)$

$= 2 \neq 0 \quad \forall x \in (-\infty, \infty)$

The function $\varphi_1, \varphi_2, \varphi_3$ are ~~linearly~~ independent.

8.e) $\varphi_1(x) = e^{ix}, \quad \varphi_2(x) = \sin x, \quad \varphi_3(x) = 2\cos x$

Soln:

$$W(\varphi_1, \varphi_2, \varphi_3) = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_1' & \varphi_2' & \varphi_3' \\ \varphi_1'' & \varphi_2'' & \varphi_3'' \end{vmatrix} = \begin{vmatrix} e^{ix} & \sin x & 2\cos x \\ ie^{ix} & \cos x & -2\sin x \\ -e^{ix} & -\sin x & -2\cos x \end{vmatrix}$$

$$= e^{ix}(-2\cos^2 x - 2\sin^2 x) - \sin x(-2ie^{ix}\cos x - 2e^{ix}\sin x) + 2\cos x(-ie^{ix}\sin x + e^{ix}\cos x)$$

$$= -2e^{ix}(\cos^2 x + \sin^2 x) + 2\sin x(ie^{ix}\cos x + e^{ix}\sin x) + 2\cos x\, e^{ix}(-i\sin x + \cos x)$$

$$= -2e^{ix} + 2ie^{ix}\sin x\cos x + 2e^{ix}\sin^2 x - 2ie^{ix}\sin x\cos x + 2e^{ix}\cos^2 x$$

$$= -2e^{ix} + 2e^{ix}(\sin^2 x + \cos^2 x)$$

$$= -2e^{ix} + 2e^{ix}$$

$W(\varphi_1, \varphi_2, \varphi_3) = 0$

$\therefore \varphi_1, \varphi_2, \varphi_3$ are linearly dependent.

8.f) $\varphi_1(x) = x, \quad \varphi_2(x) = e^{2x}, \quad \varphi_3(x) = |x|$

Soln:

$$W(\varphi_1, \varphi_2, \varphi_3) = \begin{vmatrix} x & e^{2x} & |x| \\ 1 & 2e^{2x} & 1 \\ 0 & 4e^{2x} & 0 \end{vmatrix}$$

$$= x(0 - 4e^{2x}) - e^{2x}(0) + x(4e^{2x})$$

$$= -4xe^{2x} + 4xe^{2x}$$

$$= 0$$

$\varphi_1, \varphi_2, \varphi_3$ are linearly dependent.

8.g) Consider the equation $y''' - 4y' = 0$

VQ

(a) compute three linearly independent solns

(b) compute the Wronskian of the solns found in (a)

(c) Find the φ satisfying $\varphi(0) = 0, \quad \varphi'(0) = 1, \quad \varphi''(0) = 0$

Soln:

The C.P is $r^3 - 4r = 0$

$r(r^2 - 4) = 0$

$r = 0, \quad r^2 - 4 = 0$

$r = \pm 2$

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The soln φ has the form.

$$\varphi = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$$

$$\varphi(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

where $\varphi_1(x) = 1$, $\varphi_2(x) = e^{2x}$, $\varphi_3(x) = e^{-2x}$

$+2e^{ix}\cos 2x$

$$w(\varphi_1, \varphi_2, \varphi_3) = \begin{vmatrix} 1 & e^{2x} & e^{-2x} \\ 0 & 2e^{2x} & -2e^{-2x} \\ 0 & 4e^{2x} & 4e^{-2x} \end{vmatrix}$$

$$= 1(8e^{0x} + 8e^{0x}) - e^{2x}(0) + e^{-2x}(0)$$

$$= 8 + 8 = 16$$

$$w(\varphi_1, \varphi_2, \varphi_3) = 16$$

Any soln φ is the form

$$\varphi(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x} \rightarrow (1)$$

$$\varphi(0) = c_1 + c_2 + c_3$$

$$c_1 + c_2 + c_3 = 0 \rightarrow (2)$$

$$\varphi'(x) = 2c_2 e^{2x} - 2c_3 e^{-2x}$$

$$\varphi'(0) = 2c_2 - 2c_3$$

$$2c_2 - 2c_3 = 1 \rightarrow (3)$$

$$\varphi''(x) = 4c_2 e^{2x} + 4c_3 e^{-2x}$$

$$\varphi''(0) = 4c_2 + 4c_3$$

$$4c_2 + 4c_3 = 0 \rightarrow (4)$$

$(3) \times 2 \Rightarrow \quad 4c_2 - \cancel{4c_3} = 2$

$(4) \Rightarrow \quad 4c_2 + \cancel{4c_3} = 0$

$$8c_2 = 2$$

$$c_2 = \frac{1}{4}$$

Sub in (4)

$$4\left(\frac{1}{4}\right) + 4c_3 = 0$$

$$4c_3 = -1$$

$$c_3 = \frac{-1}{4}$$

C_2, C_3 sub in ⑤

$$C_1 + \frac{1}{4} - \frac{1}{4} = 0$$

$$C_1 = 0$$

These values sub in ①

$$\varphi(x) = 0 + \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x}$$

$$= \frac{1}{4}(e^{2x} - e^{-2x}) = \frac{2}{4}\sinh 2x$$

$$\varphi(x) = \frac{1}{2}\sinh 2x$$

8.h) Consider the eqn $y^{(5)} - y^{(4)} - y' + y = 0$

(a) Compute five linearly independent solns

(b) Compute the Wronskian of the solns found in (a).

(c) Find the soln φ satisfying $\varphi(0)=1$, $\varphi'(0) = \varphi''(0) = \varphi'''(0) = \varphi^{(4)}(0) = 0$

Soln:

The characteristic polynomial is

$$r^5 - r^4 - r + 1 = 0$$

$$r^4(r-1) - (r-1) = 0$$

$$(r^4-1)(r-1) = 0$$

$$r^4 - 1 = 0, \quad r - 1 = 0$$

$$(r^2-1^2)(r^2+1^2) = 0, \quad r = 1$$

$$r^2 - 1^2 = 0, \quad r^2 + 1^2 = 0$$

$$r = \pm 1, \quad r = \pm i$$

The roots are $r = 1, \pm 1, \pm i$

$$\therefore \varphi(x) = (C_1 + C_2 x)e^x + C_4 \cos x + C_5 \sin x$$

$$\varphi(x) = C_1 e^x + C_2 x e^x + C_4 \cos x + C_5 \sin x$$

$$W(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = \begin{vmatrix} e^x & xe^x & e^{-x} & \cos x & \sin x \\ e^x & xe^x + e^x & -e^{-x} & -\sin x & \cos x \\ e^x & xe^x + 2e^x & e^{-x} & -\cos x & -\sin x \\ e^x & xe^x + 3e^x & -e^{-x} & \sin x & -\cos x \\ e^x & xe^x + 4e^x & e^{-x} & \cos x & \sin x \end{vmatrix}$$

W.K.T The Theorem,

$$W(\varphi_1,\varphi_2 \cdots \varphi_n)(x) = e^{-a(x-x_0)} \, W(\varphi_1,\varphi_2 \cdots \varphi_n)(x_0)$$

put $x_0=0$. Then $a_1=-1$

$$W(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)(0) = \begin{vmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 & -1 \\ 1 & 4 & 1 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 1 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 3 & -2 & -1 & -1 \\ 0 & 4 & 0 & 0 & 0 \end{vmatrix}$$

$$= 4 \begin{vmatrix} -2 & -1 & 1 \\ 0 & -2 & 0 \\ -2 & -1 & -1 \end{vmatrix}$$

$$= 4[-2(2+0) + 1(0) + 1(0-4)]$$

$$= 4[-2(2) - 4]$$

$$= 4[-4-4] = 4[-8]$$

$$W(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)(0) = -32$$

$$\therefore W(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)(x) = e^{x} \, W(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5)(0)$$

$$= \cancel{3e^{x}} = -32e^{x}$$

Any soln φ is of the form.

$$\varphi(x) = (c_1+c_2x)e^{x} + c_3e^{-x} + c_4\cos x + c_5\sin x \rightarrow (1)$$

$$\varphi(0) = c_1+c_3+c_4$$

$\therefore \varphi(0)=1$

$$c_1+c_3+c_4=1 \rightarrow (2)$$

$$\varphi'(x) = c_1e^{x} + c_2(xe^{x}+e^{x}) - c_3e^{-x} - c_4\sin x + c_5\cos x$$

$$\varphi'(0) = c_1+c_2-c_3+c_5=0$$

$\varphi'(0)=0$

$$c_1+c_2-c_3+c_5=0 \rightarrow (3)$$

$$\varphi''(x) = c_1e^{x} + c_2(xe^{x}+2e^{x}) + c_3e^{-x} - c_4\cos x - c_5\sin x$$

$$\varphi''(0) = c_1+2c_2+c_3-c_4$$

$\varphi''(0)=0$

$$c_1+2c_2+c_3-c_4=0$$

$\varphi'''(x) = c_1 e^{x} + c_2(x e^{x} + 3e^{x}) - c_3 e^{-x} + c_4 \sin x - c_5 \cos x$

$\varphi'''(0) = c_1 + 3c_2 - c_3 - c_5$ $\qquad \varphi'''(0) = 0$

$c_1 + 3c_2 - c_3 - c_5 = 0 \rightarrow (5)$

$\varphi^{(4)}(x) = c_1 e^{x} + c_2(x e^{x} + 4e^{x}) + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$

$\varphi^{(4)}(0) = c_1 + 4c_2 + c_3 + c_4$ $\qquad \varphi^{(4)}(0) = 0$

$c_1 + 4c_2 + c_3 + c_4 = 0 \rightarrow (6)$

Solving (5) and (3)

$c_1 + 3c_2 - c_3 - c_5 = 0$

$c_1 + c_2 - c_3 + c_5 = 0$

(−) (−) (+) (−)

$2c_2 - 2c_5 = 0$

$2c_2 = 2c_5$

$\boxed{c_2 = c_5}$

Solving (4) and (6)

$c_1 + 4c_2 + c_3 + c_4 = 0$

$c_1 + 2c_2 + c_3 - c_4 = 0$

(−) (−) (−) (+)

$2c_2 + 2c_4 = 0$

$2c_2 = -2c_4$

$\boxed{c_2 = -c_4}$

(2) $\Rightarrow c_1 + c_3 + c_4 = 1$

$c_1 + c_3 - c_2 = 1 \rightarrow (7)$

(3) $\Rightarrow c_1 + c_2 - c_3 + c_5 = 0$

$c_1 + c_2 - c_3 + c_2 = 0$

$c_1 + 2c_2 - c_3 = 0 \rightarrow (8)$

Solving (7) and (8)

$c_1 + c_3 - c_2 = 1$

$c_1 - c_3 + 2c_2 = 0$

$2c_1 + c_2 = 1 \rightarrow (9)$

continue:

(4) $\Rightarrow c_1 + 2c_2 + c_3 - c_4 = 0$

$c_1 + 2c_2 + c_3 + c_2 = 0$

$c_1 + 3c_2 + c_3 = 0 \rightarrow (10)$

(5) $\Rightarrow c_1 + 3c_2 - c_3 - c_5 = 0$

$c_1 + 3c_2 - c_3 - c_2 = 0$

$c_1 + 2c_2 - c_3 = 0 \rightarrow (11)$

Solving (10) and (11)

$c_1 + 3c_2 + c_3 = 0$

$c_1 + 2c_2 - c_3 = 0$

$2c_1 + 5c_2 = 0 \rightarrow (12)$

Solve (9) and (12)

(9) $\Rightarrow 2c_1 + c_2 = 1$

(12) $\Rightarrow 2c_1 + 5c_2 = 0$

$-4c_2 = 1$

$\boxed{c_2 = -\frac{1}{4}}$

$\therefore c_2 = -c_4$

$\boxed{c_4 = +\frac{1}{4}}$

$c_5 = c_2$

$\boxed{c_5 = -\frac{1}{4}}$

(9) $\Rightarrow$ $2C_1 + C_2 = 1$ sub $C_2 = \frac{-1}{4}$

$2C_1 - \frac{1}{4} = 1$

$2C_1 = 1 + \frac{1}{4}$

$2C_1 = \frac{5}{4}$

$\boxed{C_1 = 5/8}$

from (11) $C_1 + 2C_2 - C_3 = 0$

$5/8 + 2(-1/4) - C_3 = 0$

$5/8 - \frac{2}{4} - C_3 = 0$

$\frac{5-4}{8} - C_3 = 0$

$\frac{1}{8} - C_3 = 0$

$\boxed{C_3 = \frac{1}{8}}$

These values sub in (1)

$\varphi(x) = \frac{5}{8} e^{x} - \frac{1}{4} x e^{x} + \frac{1}{8} e^{-x} + \frac{1}{4} \cos x - \frac{1}{4} \sin x$

8.i) Find four linearly independent solns of the eqn $y^{(4)} + \lambda y = 0$ Incase (a) $\lambda = 0$, (b), $\lambda > 0$, (c) $\lambda < 0$

Soln:

Consider the eqn $y^{(4)} + \lambda y = 0$

The c.p is $r^4 + \lambda = 0$

Case (i) $\lambda = 0$,

$\Rightarrow r^4 = 0$

$\Rightarrow r = 0, 0, 0, 0$

The soln is given by

$\varphi(x) = (C_1 + C_2 x + C_3 x^2 + C_4 x^3) e^{0x}$

$\varphi(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

Case (ii) $\lambda > 0$

Put $\lambda = 4k^4$

The C.p is

$$r^4 + 4k^4 = 0 \rightarrow (1)$$

$\div r^2$

$$r^2 + 4\frac{k^4}{r^2} = 0 \rightarrow (2)$$

Take

$$t = r + \frac{2k^2}{r} \rightarrow (3)$$

$$t^2 = \left(r + \frac{2k^2}{r}\right)^2 = r^2 + 4k^2 + 4\frac{k^4}{r^2}$$

$$t^2 - 4k^2 = r^2 + 4\frac{k^4}{r^2}$$

(2) $\Rightarrow$ $t^2 - 4k^2 = 0$

$$t^2 = 4k^2$$

$$t = \pm 2k$$

$t = 2k$

(3) $\Rightarrow$ $r + \frac{2k^2}{r} = 2k$

$$r^2 + 2k^2 = 2rk$$

$$r^2 - 2rk + 2k^2 = 0$$

$$r = \frac{2k \pm \sqrt{4k^2 - 4(2k^2)}}{2}$$

$$= \frac{2k \pm \sqrt{4k^2 - 8k^2}}{2}$$

$$= \frac{2k \pm i2k}{2}$$

$$r = k \pm ik$$

$t = -2k$

(3) $\Rightarrow$ $r + \frac{2k^2}{r} = -2k$

$$r^2 + 2k^2 = -2kr$$

$$r^2 + 2kr + 2k^2 = 0$$

$$r = \frac{-2k \pm \sqrt{4k^2 - 8k^2}}{2}$$

$$= \frac{-2k \pm i2k}{2}$$

$$= -k \pm ik$$

$$\therefore \varphi(x) = e^{kx}(c_1 \cos kx + c_2 \sin kx) + e^{-kx}(c_3 \cos kx + c_4 \sin kx)$$

Case (iii) $\lambda < 0$

Put $\lambda = -k^4$, $\quad y^4 - k^4 y = 0$

The c.p is

$r^4 - k^4 = 0$

$\Rightarrow (r^2-k^2)(r^2+k^2) = 0$

$r^2 - k^2 = 0$, $r^2+k^2=0$

$r^2 = k^2$, $r^2 = -k^2$

$r = \pm k$, $r = \pm ik$

$\therefore \varphi(x) = c_1 e^{kx} + c_2 e^{-kx} + c_3 \cos kx + c_4 \sin kx$

8.j) Consider the eqn $y^{(4)} - k^4 y = 0$, where k is real constant

(a) Show that $\cos kx$, $\sin kx$, $\cosh kx$, $\sinh kx$ are solns if $k \neq 0$.

(b) S.T there are non-trivial soln φ satisfying

$\varphi(0) = 0$, $\varphi'(0) = 0$, $\varphi(1) = 0$, $\varphi'(1) = 0$

(c) Compute all non-trivial solns satisfying the condition in (b)

Soln:

Given eqn is $y^{(4)} - k^4 y = 0$

The c.p is

$r^4 - k^4 = 0$

$(r^2-k^2)(r^2+k^2) = 0$

$r^2-k^2=0$, $r^2+k^2=0$

$r^2 = k^2$, $r^2 = -k^2$

$r = \pm k$, $r = \pm ik$

The roots are $k, -k, ik, -ik$

$\therefore e^{kx}, e^{-kx}, e^{ikx}, e^{-ikx}$ are four linearly independent solns,

Since, $\cos kx = \dfrac{e^{ikx} + e^{-ikx}}{2}$.

$\sin kx = \dfrac{e^{ikx} - e^{-ikx}}{2i}$

$\cos kx$, $\sin kx$ are solns of The given equations.

Similarly,

$$\cosh kx = \frac{e^{kx} + e^{-kx}}{2}$$

$$\sinh kx = \frac{e^{kx} - e^{-kx}}{2}$$ are also solns of The given eqn.

$\therefore$ Any soln ϕ of The given eqn is written

$$\phi(x) = c_1 \cos kx + c_2 \sin kx + c_3 \cosh kx + c_4 \sinh kx$$

(b) Given $\phi(0)=0$, $\phi'(0)=0$, $\phi(1)=0$, $\phi'(1)=0$

We obtain,

$$c_1 + c_3 = 0 \rightarrow (1)$$

$$c_2 + c_4 = 0 \rightarrow (2)$$

$$c_1 \cos k + c_2 \sin k + c_3 \cosh k + c_4 \sinh k = 0 \rightarrow (3)$$

$$-c_1 \sin k + c_2 \cos k + c_3 \sinh k + c_4 \cosh k = 0 \rightarrow (4) \quad [\because k \neq 0]$$

Equations (1) to (4) will have non trivial solns for c_1, c_2, c_3, c_4 if The determinant of The coefficients vanishes,

(i)
$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cos k & \sin k & \cosh k & \sinh k \\ -\sin k & \cos k & \sinh k & \cosh k \end{vmatrix} = 0$$

on expanding directly we get,

$$\cos k \cosh k = 1 \qquad (k \neq 0)$$

(c) Solving The equations (1) to (4) for nontrivial weget,

$$c_1 = c(\sinh k - \sin k)$$

$$c_3 = -c(\sinh k - \sin k)$$

$$c_2 = -c(\cosh k - \cos k)$$

$$c_4 = c(\cosh k - \cos k)$$

where c is constant

The non-trivial solns are

$$\varphi(x) = c\,[(\cosh k - \cos k)(\sinh kx - \sin kx) - (\sinh k - \sin k)(\cosh kx - \cos kx)]$$

c being any constant.

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9.a) Find the soln of φ of the eqn which satisfies

$y''' + y'' + y' + y = ■$, $\varphi(0)=0$, $\varphi'(0)=1$, $\varphi''(0)=0$

Soln:

Given non-homogeneous eqn is

$$y''' + y'' + y' + y = 1 \rightarrow ①$$

Consider the homogeneous eqn is

$$y''' + y'' + y' + y = 0 \rightarrow ②$$

The characteristic polynomial is

$$r^3 + r^2 + r + 1 = 0$$

$$r^2(r+1) + (r+1) = 0$$

$$(r^2+1)(r+1) = 0$$

$r^2+1=0$, $\quad r+1=0$

$r^2=-1$, $\quad r=-1$

$r=\pm i$

The soln of φ has the form

$$\varphi(x) = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$$

$$= c_1 e^{-x} + e^{0x}(c_2\cos x + c_3\sin x)$$

$$\varphi(x) = c_1 e^{-x} + c_2\cos x + c_3\sin x.$$

$\varphi_1 = e^{-x}$, $\quad \varphi_2 = \cos x$, $\quad \varphi_3 = \sin x$

$\varphi_1' = -e^{-x}$ $\quad$, $\varphi_2' = -\sin x$ $\quad$, $\varphi_3' = \cos x$

$\varphi_1'' = e^{-x}$ $\quad$, $\varphi_2'' = -\cos x$ $\quad$, $\varphi_3'' = -\sin x$

Where $\varphi_1, \varphi_2, \varphi_3$ are linearly independent soln.

To find particular integral:

$$\varphi_p = u_1\varphi_1 + u_2\varphi_2 + u_3\varphi_3 \rightarrow ③$$

Also $b(x)=1$

Where u_1', u_2', u_3' are given by

$$u_1'\varphi_1 + u_2'\varphi_2 + u_3'\varphi_3 = 0$$

$$u_1'\varphi_1' + u_2'\varphi_2' + u_3'\varphi_3' = 0$$

$$u_1'\varphi_1'' + u_2'\varphi_2'' + u_3'\varphi_3'' = b(x)$$

Which in this case reduce to

$$\left.\begin{array}{l} u_1'e^{-x} + u_2'\cos x + u_3'\sin x = 0 \\ -u_1'e^{-x} - u_2'\sin x + u_3'\cos x = 0 \\ u_1'e^{-x} - u_2'\cos x - u_3'\sin x = 1 \end{array}\right\} \rightarrow (4)$$

The determinant of the coefficient of (4) is

$$W(\varphi_1,\varphi_2,\varphi_3)(x) = \begin{vmatrix} e^{-x} & \cos x & \sin x \\ -e^{-x} & -\sin x & \cos x \\ e^{-x} & -\cos x & -\sin x \end{vmatrix} \rightarrow (5)$$

W.k.T

$$W(\varphi_1,\varphi_2,\varphi_3)(x) = e^{-a_1(x-x_0)} W(\varphi_1,\varphi_2,\varphi_3)(0) \rightarrow (6)$$

using (1) $a_1 = 1, \quad x_0 = 0$

$$W(\varphi_1,\varphi_2,\varphi_3)(x_0) = W(\varphi_1,\varphi_2,\varphi_3)(0)$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1[0+1] - 1[0-1] + 0$$

$$= 1+1$$

$$W(\varphi_1,\varphi_2,\varphi_3)(x_0) = 2 \neq 0$$

$$W(\varphi_1,\varphi_2,\varphi_3)(x) = e^{-1(x-0)} \cdot 2$$

$$W(\varphi_1,\varphi_2,\varphi_3)(x) = 2e^{-x}$$

W.k.T

$$u_k(x) = \int \frac{W_k(x)\, b(x)}{W(x)}\, dx, \quad k=1,2\ldots n$$

Now, $$u_1'(x) = \frac{W_1(x)\, b(x)}{W(x)}$$

Where w_1 is obtained from w by replacing 1st column $(0,0,1)$

using ⑤

$$u_1'(x) = \frac{\begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix}(1)}{2e^{-x}}$$

$$= \frac{0 - \cos x\,[0 - \cos x] + \sin x\,[0 + \sin x]}{2e^{-x}}$$

$$u_1'(x) = \frac{\cos^2 x + \sin^2 x}{2e^{-x}} = \frac{e^{x}}{2}$$

$$u_1(x) = \int \frac{e^{x}}{2}\,dx$$

$$u_1(x) = \frac{e^{x}}{2}$$

Similarly $\quad u_2'(x) = \dfrac{w_2(x)\,b(x)}{w(x)}$

Where w_2 is obtained from w by replacing 2nd column

$(0,0,1)$ using ⑤

$$u_2'(x) = \frac{\begin{vmatrix} e^{-x} & 0 & \sin x \\ -e^{-x} & 0 & \cos x \\ e^{-x} & 1 & -\sin x \end{vmatrix}(1)}{2e^{-x}}$$

$$= \frac{e^{-x}[0 - \cos x] - 0 + \sin x[-e^{-x}]}{2e^{-x}}$$

$$= \frac{-e^{-x}\cos x - e^{-x}\sin x}{2e^{-x}} = \frac{-e^{-x}(\cos x + \sin x)}{2e^{-x}}$$

$$u_2'(x) = -\frac{(\cos x + \sin x)}{2}$$

$$u_2(x) = \frac{-1}{2}\int(\cos x + \sin x)\,dx$$

$$= \frac{-1}{2}[\sin x - \cos x]$$

$$u_2(x) = \frac{1}{2}[\cos x - \sin x]$$

$$u_3'(x) = \frac{W_3(x)\, b(x)}{W(x)}$$

Where W_3 is obtained from W by replacing 3^{rd} column $(0,0,1)$ using ⑤

$$u_3'(x) = \frac{\begin{vmatrix} e^{-x} & \cos x & 0 \\ -e^{-x} & -\sin x & 0 \\ e^{-x} & -\cos x & 1 \end{vmatrix} (1)}{2e^{-x}}$$

$$= \frac{e^{-x}[-\sin x] - \cos x[-e^{-x}]}{2e^{-x}}$$

$$= \frac{e^{-x}(-\sin x + \cos x)}{2e^{-x}}$$

$$u_3'(x) = \frac{1}{2}(-\sin x + \cos x)$$

$$u_3(x) = \frac{1}{2}\int(-\sin x + \cos x)\,dx$$

$$= \frac{1}{2}[\cos x + \sin x]$$

$$u_3(x) = \frac{\cos x + \sin x}{2}$$

A Particular soln of ④ is

$$\psi_p = \frac{e^{x}}{2}\cdot e^{-x} + \frac{1}{2}(\cos x - \sin x)\cos x + \frac{1}{2}(\cos x + \sin x)\sin x$$

$$= \frac{1}{2} + \frac{\cos^2 x}{2} - \frac{\sin x\cos x}{2} + \frac{\sin x \cos x}{2} + \frac{\sin^2 x}{2}$$

$$= \frac{1}{2} + \frac{1}{2}(\cos^2 x + \sin^2 x) = \frac{1}{2} + \frac{1}{2}$$

$$\psi_p = 1$$

The General soln of 4 of is the form,

$$\psi(x) = \psi_p + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3$$

$$\psi(x) = 1 + c_1 e^{-x} + c_2\cos x + c_3 \sin x \rightarrow ⑦$$

$$\psi'(x) = -c_1 e^{-x} - c_2\sin x + c_3\cos x$$

$$\psi''(x) = c_1 e^{-x} - c_2\cos x - c_3\sin x$$

$$\psi(0) = 1 + c_1 + c_2$$

$\psi(0) = 0$

$$1 + c_1 + c_2 = 0$$

$y'(0) = -c_1 + c_3$

$-c_1 + c_3 = 1 \rightarrow (9)$

$y''(0) = c_1 - c_2$

$c_1 - c_2 = 0 \rightarrow (10)$

Solve (8) and (10)

$(8) \Rightarrow \quad c_1 + c_2 = -1$

$(10) \Rightarrow \quad c_1 - c_2 = 0$

$2c_1 = -1$

$c_1 = -1/2$

$(10) \Rightarrow \quad -\frac{1}{2} - c_2 = 0$

$c_2 = -1/2$

$(9) \Rightarrow \quad -c_1 + c_3 = 1$

$-(-1/2) + c_3 = 1$

$c_3 = 1 - 1/2$

$c_3 = 1/2$

These values sub in (7)

$y(x) = 1 - \frac{1}{2} e^{-x} - \frac{1}{2} \cos x + \frac{1}{2} \sin x$